Partial Fractions

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1 Partial fractions and the coverup method

1.1 Heaviside Cover-up Method

1.1.1 Introduction

The cover-up method was introduced by Oliver Heaviside as a fast way to do a decomposition into partial fractions. This is an essential step in using the Laplace transform to solve differential equations, and this was more or less Heaviside's original motivation.

The cover-up method can be used to make a partial fractions decomposition of a proper rational function $\frac{P(s)}{Q(s)}$ *whenever the denominator can be factored into distinct linear factors.* We will see that we can extend the coverup method to also deal with distinct quadratic factors.

Note. We put this section first, because the coverup method is so useful and many people have not seen it. Some of the later examples rely on the full algebraic method of undetermined coefficients presented in the next section. If you have never seen partial fractions you should read that section first.

1.1.2 Linear Factors

We first show how the method works on a simple example, and then show why it works.

Example PF.1. Decompose $\frac{s-7}{(s-1)(s+2)}$ into partial fractions.

Solution: We know the answer will have the form

$$\frac{s-7}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}. \tag{1}$$

To determine A by the cover-up method, on the left-hand side we mentally remove (or cover up with a finger) the factor $s-1$ associated with A, and substitute $s = 1$ into what's left; this gives A:

$$\left.\frac{s-7}{\qquad (s+2)}\right|_{s=1} = \frac{1-7}{1+2} = -2 = A. \tag{2}$$

Similarly, B is found by covering up the factor $s+2$ on the left, and substituting $s = -2$ into what's left. This gives

$$\left.\frac{s-7}{(s-1)\qquad}\right|_{s=-2} = \frac{-2-7}{-2-1} = 3 = B.$$

Thus, our answer is

$$\frac{s-7}{(s-1)(s+2)} = \frac{-2}{s-1} + \frac{3}{s+2}. \tag{3}$$

1.1.3 Why does the method work?

The reason is simple. The "right" way to determine A from Equation 1 would be to multiply both sides by $(s-1)$; this would give

$$\frac{s-7}{(s+2)} = A + \frac{B}{s+2}(s-1). \tag{4}$$

Now if we substitute $s = 1$, what we get is exactly Equation 2, since the term on the right with B disappears. The cover-up method therefore is just an easy and efficient way of doing the calculations.

In general, if the denominator of the proper rational function factors into the product of distinct linear factors:

$$\frac{P(s)}{(s-a_1)(s-a_2)\cdots(s-a_r)} = \frac{A_1}{s-a_1} + \ldots + \frac{A_r}{s-a_r}, \quad a_i \neq a_j ,$$

then A_i is found by covering up the factor $s - a_i$ on the left, and setting $s = a_i$ in the rest of the expression.

Example PF.2. Decompose $\frac{1}{s^3-s}$ into partial fractions.

Solution: Factoring, $s^3 - s = s(s^2-1) = s(s-1)(s+1)$. By the cover-up method,

$$\frac{1}{s(s-1)(s+1)} = \frac{-1}{s} + \frac{1/2}{s-1} + \frac{1/2}{s+1}.$$

To be honest, the real difficulty in all of the partial fractions methods (the cover-up method being no exception) is in factoring the denominator.

1.1.4 Quadratic Factors

There are multiple ways to deal with a quadratic factors that don't have real roots.

1. Factor the quadratic factor into complex linear factors.
2. Use 'complex coverup'
3. Use a algebraic techniques with just real numbers

We will show all three methods using the same example

Example PF.3. Decompose $G(s) = \frac{s-1}{(s+1)(s^2+4)}$ by factoring the denominator into complex linear terms

Solution: $\frac{s-1}{(s+1)(s^2+4)} = \frac{s-1}{(s+1)(s-2i)(s+2i)} = \frac{A}{s+1} + \frac{B}{s-2i} + \frac{C}{s+2i}$.

Using coverup we see that $A = -2/5$, $B = \frac{2i-1}{(2i+1)(4i)} = \frac{4-3i}{20}$, $C = \frac{4+3i}{20}$. So,

$$G(s) = \frac{-2/5}{s+1} + \frac{(4-3i)/20}{s-2i} + \frac{(4+3i)/20}{s+2i} = \frac{-2/5}{s+1} + \frac{(8s+12)/20}{s^2+4} = \frac{-2/5}{s+1} + \frac{(2s/5+3/5)}{s^2+4}.$$

Example PF.4. Decompose $G(s) = \frac{s-1}{(s+1)(s^2+4)}$ without using complex techniques.

Solution: Notice that in the previous example in the last expression for $G(s)$ the numerator of the s^2+4 term in the partial fraction decomposition is a linear term instead of a constant. This is the general rule for quadratic terms.

$$\frac{s-1}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}. \tag{5}$$

Notice the quadratic factor gets a linear term in the numerator. Notice also that the number of unknown coefficients is the same as the degree of the denominator in the original fraction.

Using coverup we find $A = -2/5$. Now we can do some algebra to find B and C. Muliplying Equation 5 through by the denominator gives

$$s-1 = A(s^2+4) + (Bs+C)(s+1) = (A+B)s^2 + (B+C)s + (4A+C).$$

Equate the coefficients on both sides:

$$\begin{aligned} s^2: &\quad 0 = A+B \\ s: &\quad 1 = B+C \\ s^2: &\quad -1 = 4A+C \end{aligned}$$

Since we already know A, the first equation gives $B = 2/5$, and then the second gives $C = 3/5$. Comparing this with the previous example we see we've found the same answer.

Example PF.5. Decompose $\dfrac{s-1}{(s+1)(s^2+4)}$ using 'complex coverup'.

Solution: This starts the same as the previous example.

$$\frac{s-1}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}. \tag{6}$$

Coverup gives us $A = -2/5$. To do complex coverup, mentally multiply the above equation by s^2+4 and then substitute $s = 2i$. We get

$$\frac{2i-1}{2i+1} = B(2i) + C \;\Leftrightarrow\; \frac{3+4i}{5} = C + 2Bi.$$

So, $C = 3/5$ and $B = 2/5$. (Same as before.)

Example PF.6. Don't be fooled by quadratic terms that factor into linear ones.

$$\frac{1}{(s+1)(s^2-4)} = \frac{1}{(s+1)(s+2)(s-2)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-2}.$$

Example PF.7. Don't forget that the rational function must be proper. For example, decompose $\dfrac{s^3+2s+1}{s^2+s-2}$ using partial fractions.

Solution: First, we must use long-division to make this proper.

$$\frac{s^3+2s+1}{s^2+s-2} = s-1+\frac{5s-1}{s^2+s-2} = s-1+\frac{5s-1}{(s+2)(s-1)} = s-1+\frac{A}{s+2}+\frac{B}{s-1}.$$

Using coverup we get $A = 11/3$, $B = 4/3$.

1.1.5 Repeated Linear Factors

The cover-up method can also be used if a linear factor is repeated, but it gives just partial results. It applies only to the *highest power of the linear factor*.

Example PF.8. Decompose $\dfrac{1}{(s-1)^2(s+2)}$.

Solution: We write

$$\frac{1}{(s-1)^2(s+2)} = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s+2}. \tag{7}$$

To find A cover up $(s-1)^2$ and set $s = 1$; you get $A = 1/3$. To find C, cover up $s+2$, and set $s = -2$; you get $C = 1/9$.

This leaves B which cannot be found by the cover-up method. But since A and C are already known in Equation 7, B can be found by substituting any numerical value (other than 1 or -2) for s in (7). For instance, if we put $s = 0$ and remember that $A = 1/3$ and $C = 1/9$, we get

$$\frac{1}{2} = \frac{1/3}{1} + \frac{B}{-1} + \frac{1/9}{2},$$

giving $B = -1/9$.

B could also be found by applying the method of undetermined coefficients to the Equation 7; note that since A and C are known, it is enough to get a single linear equation in order to determine B — simultaneous equations are no longer needed.

The fact that the cover-up method works for just the *highest power* of the repeated linear factor can be seen just as before. In the above example for instance, the cover-up method for finding A is just a short way of multiplying Equation 9 through by $(s-1)^2$ and then substituting $s = 1$ into the resulting equation.

1.2 Partial Fractions: Undetermined Coefficients

1.2.1 Introduction

Logically this section should precede the previous one on coverup since it explains what we are doing with partial fractions and shows an algebraic method that never fails. However, since most students in this course will have seen partial fractions before it seemed reasonable to start with the coverup method.

1.2.2 Rational Functions

A **rational function** is one that is the ratio of two polynomials. For example

$$\frac{s+1}{s^2+7s+9} \quad \text{and} \quad \frac{s^2+7s+9}{s+1}$$

are both rational functions.

A rational function is called **proper** if the degree of the numerator is strictly smaller than the degree of the denominator; in the examples above, the first is proper while the second is not.

Long-division: Using long-division we can always write an improper rational function as a polynomial plus a proper rational function. The partial fraction decomposition only applies to proper functions.

Example PF.9. Use long-division to write $\frac{s^3+2s+1}{s^2+s-2}$ as the sum of a polynomial and a proper rational function.

Solution:

$$\begin{array}{r|rrrr} & s-1 & & & \\ \hline s^2+s-2 & s^3 & + & 2s & +1 \\ & s^3 & +s^2 & -2s & \\ \hline & & -s^2 & +4s & +1 \\ & & -s^2 & -s & +2 \\ \hline & & & 5s & -1 \end{array}$$

Therefore,

$$\frac{s^3+2s+1}{s^2+s-2} = s-1+\frac{5s-1}{s^2+s-2}.$$

1.2.3 Linear Factors

Here we assume the denominator factors into distinct linear factors. We start with a simple example. We will explain the general principle immediately afterwords.

Example PF.10. Decompose $R(s) = \frac{s-3}{(s-2)(s-1)}$ using partial fractions.

Solution:

$$\frac{s-3}{(s-2)(s-1)} = \frac{A}{s-2} + \frac{B}{s-1}.$$

Multiplying both sides by the denominator on the left gives

$$s-3 = A(s-1) + B(s-2) \tag{8}$$

The sure algebraic way is to expand out the right hand side and equate the coefficients with those of the polynomial on the left.

$$s-3 = (A+B)s + (-A-2B) \Rightarrow \begin{cases} \text{coeff. of } s: & 1 = A+B \\ \text{coeff. of } 1: & -3 = -A-2B \end{cases}$$

We solve this system of equations to find the undetermined coefficients A and B: $A=-1$, $B=2$.

Answer: $R(s) = -1/(s-2) + 2/(s-1)$.

Note. Not surprisingly, using the coverup method by plugging the roots of each factor into Equation 8 would be easier because when you do this every term except one becomes 0.

$$\begin{array}{ll} \text{Plug in } s=1 & \Rightarrow -2 = B(-1) \Rightarrow B=2 \\ \text{Plug in } s=2 & \Rightarrow -1 = A(1) \Rightarrow A=-1. \end{array}$$

In general, if $P(s)/Q(s)$ is a proper rational function and $Q(s)$ factors into distinct linear factors $Q(s) = (s-a_1)(s-a_2)\cdots(s-a_n)$ then

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \cdots + \frac{A_n}{s-a_n}.$$

The proof of this is not hard, but we will not give it. Remember you must have a *proper* rational function and each of the factors must be distinct. Repeated factors are discussed below.

Example PF.11. Decompose $\frac{3}{s^3 - 3s^2 - s + 3}$.

Solution: The hardest part of this problem is to factor the denominator. For higher order polynomials this might be impossible. In this case you can check

$$s^3 - 3s^2 - s + 3 = (s-1)(s+1)(s-3).$$

The partial fractions decomposition is

$$\frac{3}{(s-1)(s+1)(s-3)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s-3}.$$

Multiplying through by the denominator gives

$$3 = A(s+1)(s-3) + B(s-1)(s-3) + C(s-1)(s+1).$$

Plugging in $s = 1$ gives $A = -3/4$, likewise $s = -1$ gives $B = 3/2$ and $s = 3$ gives $C = -3/4$.

1.2.4 Quadratic Factors

Example PF.12. Decompose $\frac{5s+6}{(s^2+4)(s-2)}$.

Solution: We write

$$\frac{5s+6}{(s^2+4)(s-2)} = \frac{As+B}{s^2+4} + \frac{C}{s-2}. \qquad (9)$$

We first determine C by the cover-up method, getting $C = 2$. Then A and B can be found by the method of undetermined coefficients; the work is greatly reduced since we need to solve only two simultaneous equations to find A and B, not three.

Following this plan, using $C = 2$, we combine terms on the right of (9) so that both sides have the same denominator. The numerators must then also be equal, which gives us

$$5s + 6 = (As + B)(s-2) + 2(s^2+4). \qquad (10)$$

Comparing the coefficients of s^2 and of the constant terms on both sides of (10) gives the two equations

$$0 = A + 2 \qquad \text{and} \qquad 6 = -2B + 8,$$

from which $A = -2$ and $B = 1$.

In using Equation 10, one could have instead compared the coefficients of s, getting $5 = -2A + B$, leading to the same result, but providing a valuable check on the correctness of the computed values for A and B.

In Example PF.12, an alternative to undetermined coefficients would be to substitute two numerical values for s into the original Equation 9, say $s = 0$ and $s = 1$ (any values other than $s = 2$ are usable). Again one gets two simultaneous equations for A and B. This method requires addition of fractions, and is usually better when only one coefficient remains to be determined (as in the example just below).

1.2.5 Repeated Linear Factors

For repeated linear factors we need one partial fraction term for each power of the factor as illustrated by the following example.

Example PF.13. Decompose $G(s) = \dfrac{2s}{s^3(s+1)^2(s+2)}$ using partial fractions.

Solution:

$$G(s) = \frac{2s}{s^3(s+1)^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1} + \frac{E}{(s+1)^2} + \frac{F}{s+2}$$

Here the denominator has a linear factor s repeated three times (term s^3), and a linear factor $(s+1)$ repeated twice (term $(s+1)^2$); hence three partial fractions are associated with the first, while two are associated with the latter. The term (s+2) which is not repeated leads to one partial fraction as previously seen. You can check that the coefficients are

$$A = -5/2,\ B = 1,\ C = 0,\ D = 2,\ E = 2,\ F = 1/2.$$

1.2.6 Repeated Quadratic Factors

Just like repeated linear factors, quadratic factors have one term for each power of the factor as illustrated in the following example.

Example PF.14. Find $G(s) = \dfrac{2s}{s(s^2+1)^2(s^2+4s+2)}$ using partial fractions.

Solution: The partial fractions decomposition is

$$G(s) = \frac{2s}{s(s^2+1)^2(s^2+4s+6)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} + \frac{Ds+E}{(s^2+1)^2} + \frac{Fs+G}{s^2+4s+6}$$

Note the repeated factor $(s^2+1)^2$ lead to two partial fraction terms.

We won't compute the coefficients –you can do this by going through the algebra.

1.2.7 Complex Factors

As in Example PF.3 we can allow complex roots. In this case all quadratic terms factor into linear terms.

Example PF.15. Decompose $s/(s^2+\omega^2)$ using complex partial fractions.

Solution:

$$\frac{s}{s^2+\omega^2} = \frac{s}{(s-i\omega)(s+i\omega)} = \frac{A}{s-i\omega} + \frac{B}{s+i\omega}.$$

Multiplying through by the denominator gives $s = A(s+i\omega) + B(s-i\omega)$.

Plug in $s = i\omega \Rightarrow A = 1/2$.

Plug in $s = -i\omega \Rightarrow B = 1/2$.